Composite-particles (Boson, Fermion) Theory of Fractional Quantum Hall Effect

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Abstract

A quantum statistical theory is developed for a fractional quantum Hall effects in terms of composite bosons (fermions) each of which contains a conduction electron and an odd (even) number of fluxons. The cause of the QHE is by assumption the phonon exchange attraction between the conduction electron ("electron", "hole") and fluxons (quanta of magnetic fluxes). We postulate that c-fermions with *any* even number of fluxons have an effective charge (magnitude) equal to the electron charge e. The density of c-fermions with m fluxons, $n_\phi^{(m)}$, is connected with the electron density n_e by $n_\phi^{(m)} = n_e/m$, which implies a more difficult formation for higher m, generating correct values me^2/h for the Hall conductivity $\sigma_H \equiv j/E_H$. For condensed c-bosons the density of c-bosons-with-m fluxons, $n_\phi^{(m)}$, is connected with the boson density n_0 by $n_\phi^{(m)} = n_0/m$. This yields $\sigma_H = m\,e^2/h$ for the magnetoconductivity, the value observed of the QHE at filling factor $\nu = 1/m$ (m =odd numbers). Laughlin's theory and results about the fractional charge are not borrowed in the present work.

Keywords: fractional QHE, fluxon, c-particle, effective charge, filling factor

1. Introduction

In 1983 Laughlin introduced a Laughlin ground-state wave function for a system of N electrons at $\nu = 1/3$, and studied the elementary excitations (quasiparticles) over the ground-state [1]. He predicted that the quasiparticle

has a *fractional charge*:

$$e^* = e/3 \, . \tag{1}$$

This prediction was later confirmed in the magnetotransport experiments by Clark *et al.* and others [2]. The 1998 Nobel prize was shared by Tsui, Störmer (experimental discovery) and Laughlin (theory) for their contribution to the fractional QHE.

The prevalent theories [4] based on the Laughlin wave function in the Schrödinger picture deal with the QHE at 0 K and immediately above. The system ground-state, however, does not carry a current. To interpret the experimental data it is convenient to introduce composite (c-) particles (bosons, fermions). The c-boson (fermion), each containing an electron and an odd (even) number of *flux quanta (fluxons)*, were introduced by Zhang *et al.* [5] and others (Jain [6]) for the description of the fractional QHE (Fermi liquid). All QHE states with distinctive plateaus in ρ_{H} are observed below the critical temperature $T_{\mathrm{c}} \simeq 0.5 \, \mathrm{K}$. The QHE in graphene, a single sheet of graphite, is an exception. It is desirable to treat the QHE below and above T_{c} in a unified manner. The extreme accuracy (precision $\sim 10^{-8}$) in which each plateau is observed means that the current density j must be computed exactly without averaging. In the prevalent theories [4] the electron-electron interaction and Pauli's exclusion principle are regarded as the cause for the QHE. Both are essentially repulsive and cannot account for the fact that the c-particles are bound, that is, they are in negative-energy states. Besides, the prevalent theories have limitations:

- The zero temperature limit is taken at the outset. Then the question why QHE is observed below $0.5 \, \mathrm{K}$ in GaAs/AlGaAs cannot be answered. We better have a theory for all temperatures.

- The high-field limit is taken at the outset. The integer QHE are observed for small integer P only. The question why the QHE is not observed for high P (weak field) cannot be answered. We better describe the phenomena for all fields.

- The Hall resistivity ρ_{H} value $(Q/P)(h/e^2)$ is obtained in a single stroke. To obtain ρ_{H} we need two separate measurements of the Hall field E_{H} and the current density j. We must calculate (E_{H}, j) and take the ratio $E_{\mathrm{H}}/j = \rho_{\mathrm{H}}$.

The main purpose of the present work is to show that the known fractional QHE can be described within the frame-work of c-particles model without using Laughlin's theory and results about the fractional charge.

2. The Hamiltonian

Fujita and Okamura developed a quantum statistical theory of the QHE [6]. We follow this theory. See this reference for more details.

There is a remarkable similarity between the QHE *and* the High-Temperature Superconductivity (HTSC), both occurring in two-dimensional (2D) systems as pointed out by Laughlin [7]. The major superconducting properties observed in the HTSC are (a) zero resistance, (b) a sharp phase change, (c) an energy gap below T_c, (d) flux quantization, (e) Meissner effect, and (f) Josephson effects. All these have been observed in GaAs/AlGaAs. We regard the *phonon exchange attraction* as the causes of both QHE and superconductivity. Starting with a reasonable Hamiltonian, we calculate everything using quantum statistical mechanics.

The countability concept of the fluxons, known as the *flux quantization*:

$$B = \frac{N_\phi}{A}\frac{h}{e} \equiv n_\phi \frac{h}{e}\,, \tag{2}$$

where A = sample area, N_ϕ = fluxon number (integer) and h = Planck constant, is originally due to Onsager [8]. The magnetic (electric) field is an axial (polar) vector and the associated fluxon (photon) is a half-spin fermion (full-spin boson). The magnetic (electric) flux line cannot (can) terminate at a sink, which supports the fermionic (bosonic) nature of the associated fluxon (photon). No half-spin fermion can annihilate by itself because of angular momentum conservation. The electron spin originates in the relativistic quantum equation (Dirac's theory of electron) [9]. The discrete (two) quantum numbers ($\sigma_z = \pm 1$) cannot change in the continuous limit, and hence the spin must be conserved. The countability and statistics of the fluxon are fundamental particle properties. We postulate that *the fluxon is a half-spin fermion with zero mass and zero charge.*

We assume that the magnetic field $\mathbf{B}$ is applied perpendicular to the interface. The 2D Landau level energy,

$$\varepsilon = \hbar\omega_c \left(N_L + \frac{1}{2} \right)\,, \quad \omega_c \equiv eB/m^*\,, \tag{3}$$

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with the states (N_{L}, k_y), $N_{\mathrm{L}} = 0, 1, 2, \cdots$, have a great degeneracy. The cyclotron frequency ω_{c} contains the electron effective mass m^*. The Center-of-Mass (CM) of *any* c-particle moves as a fermion (boson). The eigenvalues of the CM momentum are limited to 0 or 1 (unlimited) if it contains an odd (even) number of elementary fermions. This rule is known as the *Ehrenfest-Oppenheimer-Bethe's* (EOB's) *rule* [10]. Hence the CM motion of the composite containing an electron and Q fluxons is bosonic (fermionic) if Q is odd (even). The system of c-bosons condenses below the critical temperature T_{c} and exhibits a superconducting state while the system of c-fermions shows a Fermi liquid behavior.

A longitudinal phonon, acoustic or optical, generates a density wave, which affects the electron (fluxon) motion through the charge displacement (current). The exchange of a phonon between electrons and fluxons generates an *attractive* transition.

Bardeen, Cooper and Schrieffer (BCS) [11] assumed the existence of Cooper pairs [12] in a superconductor, and wrote down a Hamiltonian containing the "electron" and "hole" kinetic energies and the pairing interaction Hamiltonian with the phonon variables eliminated. We start with a BCS-like Hamiltonian $\mathcal{H}$ for the QHE [6]:

$$
\begin{aligned}
\mathcal{H} = {\sum_{\mathbf{k}}}' \sum_s \varepsilon_{\mathbf{k}}^{(1)} n_{\mathbf{k}s}^{(1)} + {\sum_{\mathbf{k}}}' \sum_s \varepsilon_{\mathbf{k}}^{(2)} n_{\mathbf{k}s}^{(2)} + {\sum_{\mathbf{k}}}' \sum_s \varepsilon_{\mathbf{k}}^{(3)} n_{\mathbf{k}s}^{(3)} - {\sum_{\mathbf{q}}}' {\sum_{\mathbf{k}}}' \\
{\sum_{\mathbf{k}'}}' \sum_s v_0 \left[B_{\mathbf{k}'\mathbf{q}s}^{(1)\dagger} B_{\mathbf{k}\mathbf{q}s}^{(1)} + B_{\mathbf{k}'\mathbf{q}s}^{(1)\dagger} B_{\mathbf{k}\mathbf{q}s}^{(2)\dagger} + B_{\mathbf{k}'\mathbf{q}s}^{(2)} B_{\mathbf{k}\mathbf{q}s}^{(1)} + B_{\mathbf{k}'\mathbf{q}s}^{(2)} B_{\mathbf{k}\mathbf{q}s}^{(2)\dagger} \right],
\end{aligned} \tag{4}
$$

where $n_{\mathbf{k}s}^{(j)} = c_{\mathbf{k}s}^{(j)\dagger} c_{\mathbf{k}s}^{(j)}$ is the number operator for the "electron" (1) ["hole" (2), fluxon (3)] at momentum $\mathbf{k}$ and spin s with the energy $\varepsilon_{\mathbf{k},s}^{(j)}$, with annihilation (creation) operators c ($c^\dagger$) satisfying the Fermi anti-commutation rules:

$$
\{c_{\mathbf{k}s}^{(i)}, c_{\mathbf{k}'s'}^{(j)\dagger}\} \equiv c_{\mathbf{k}s}^{(i)} c_{\mathbf{k}'s'}^{(j)\dagger} + c_{\mathbf{k}'s'}^{(j)\dagger} c_{\mathbf{k}s}^{(i)} = \delta_{\mathbf{k},\mathbf{k}'} \delta_{s,s'} \delta_{i,j}, \quad \{c_{\mathbf{k}s}^{(i)}, c_{\mathbf{k}'s'}^{(j)}\} = 0. \tag{5}
$$

The fluxon number operator $n_{\mathbf{k}s}^{(3)}$ is represented by $a_{\mathbf{k}s}^\dagger a_{\mathbf{k}s}$ with a ($a^\dagger$) satisfying the anti-commutation rules:

$$
\{a_{\mathbf{k}s}, a_{\mathbf{k}'s'}^\dagger\} = \delta_{\mathbf{k},\mathbf{k}'} \delta_{s,s'}, \quad \{a_{\mathbf{k}s}, a_{\mathbf{k}'s'}\} = 0. \tag{6}
$$

The phonon exchange can create electron-fluxon composites, bosonic or fermionic, depending on the number of fluxons. The center-of-mass of any

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composite moves as a fermion (boson) if it contains an odd (even) numbers of elementary fermions. We call the conduction-electron composite with an odd (even) number of fluxons c-boson (c-fermion). The electron (hole)-type c-particles carry negative (positive) charge. We expect that electron (hole)-type c-bosons are generated by the phonon-exchange attraction. The pair operators B are defined by

$$B^{(1)\dagger}_{\mathbf{kq},s} \equiv c^{(1)\dagger}_{\mathbf{k+q}/2,s} a^{\dagger}_{-\mathbf{k+q}/2,-s}, \quad B^{(2)}_{\mathbf{kq},s} \equiv a_{-\mathbf{k+q}/2,-s} c^{(2)}_{\mathbf{k+q}/2,s}. \tag{7}$$

The prime on the summation in Eq. (4) means the restriction: $0 < \varepsilon^{(j)}_{\mathbf{k}s} < \hbar\omega_{\mathrm{D}}$, $\omega_{\mathrm{D}} = $ Debye frequency. The pairing interaction terms in Eq. (4) conserve the charge. The term $-v_0 B^{(1)\dagger}_{\mathbf{k'q}s} B^{(1)}_{\mathbf{kq}s}$, where $v_0 \equiv |V_{\mathbf{q}} V'_{\mathbf{q}}| (\hbar\omega_0 A)^{-1}$, $A = $ sample area, is the pairing strength, generates a transition in electron-type c-fermion states. Similarly, the exchange of a phonon generates a transition between hole-type c-fermion states, represented by $-v_0 B^{(2)\dagger}_{\mathbf{k'q}s} B^{(2)\dagger}_{\mathbf{kq}s}$. The phonon exchange can also pair-create (pair-annihilate) electron (hole)-type c-boson pairs, and the effects of these processes are represented by $-v_0 B^{(1)\dagger}_{\mathbf{k'q}s} B^{(2)\dagger}_{\mathbf{kq}s}$ $\left(-v_0 B^{(1)}_{\mathbf{kq}s} B^{(2)}_{\mathbf{kq}s}\right)$.

The Cooper pair is formed from two "electrons" (or "holes"). Likewise the c-bosons may be formed by the phonon-exchange attraction from c-fermions and fluxons. If the density of the c-bosons is high enough, then the c-bosons will be condensed and exhibit a superconductivity.

To treat superconductivity we modify the pair operators in Eq. (7) as

$$B^{(1)\dagger}_{\mathbf{kq},s} \equiv c^{(1)\dagger}_{\mathbf{k+q}/2,s} c^{(1)\dagger}_{-\mathbf{k+q}/2,-s}, \quad B^{(2)}_{\mathbf{kq},s} \equiv c^{(2)}_{-\mathbf{k+q}/2,-s} c^{(2)}_{\mathbf{k+q}/2,s}. \tag{8}$$

Then, the pairing interaction terms in Eq. (4) are formally identical with those in the generalized BCS Hamiltonian [13].

We first consider integer QHE. We choose a conduction electron and a fluxon for the pair. The c-bosons, having the linear dispersion relation, can move in all directions in the plane with the constant speed $(2/\pi)v^{(j)}_{\mathrm{F}}$. The supercurrent is generated by $\mp$ c-bosons monochromatically condensed, running along the sample length. The supercurrent density (magnitude) j, calculated by the rule: $j = $ (carrier charge e^*) $\times$ (carrier density n_0) $\times$ (drift velocity v_{d}), is given by

$$j \equiv e^* n_0 v_{\mathrm{d}} = e^* n_0 \frac{2}{\pi} \left| v^{(1)}_{\mathrm{F}} - v^{(2)}_{\mathrm{F}} \right|. \tag{9}$$

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The induced Hall field (magnitude) E_{H} equals $v_{\mathrm{d}}B$. The magnetic flux is quantized:

$$B = n_\phi \Phi_0 \,, \tag{10}$$

where $n_\phi \equiv N_\phi/A$ is the fluxon density and $\Phi_0 (\equiv e/h)$ is the magnetic flux quantum. Hence we obtain

$$\rho_{\mathrm{H}} \equiv \frac{E_{\mathrm{H}}}{j} = \frac{v_{\mathrm{d}}B}{e^* n_0 v_{\mathrm{d}}} = \frac{1}{e^* n_0} n_\phi \left(\frac{h}{e}\right) \,. \tag{11}$$

We assume that *the c-fermion containing an electron and an even number of fluxons has a charge magnitude e*. For the integer QHE, $e^* = e$, $n_\phi = n_0$ for the carriers, thus we obtain $\rho_{\mathrm{H}} = h/e^2$, the correct plateau value observed for the principal QHE at $\nu = 1$.

The supercurrent generated by equal numbers of $\mp$ c-bosons condensed monochromatically is neutral. This is reflected in our calculations in Eq. (9). The supercondensate whose motion generates the supercurrent must be neutral. If it has a charge, it would be accelerated indefinitely by the external field because the impurities and phonons cannot stop the supercurrent to grow. That is, the circuit containing a superconducting sample and a battery must be burnt out if the supercondensate is not neutral. In the calculation of ρ_{H} in Eq. (11), we used the *unaveraged* drift velocity difference $(2/\pi)|v_{\mathrm{F}}^{(1)} - v_{\mathrm{F}}^{(2)}|$, which is significant. *Only* the unaveraged drift velocity cancels out v_{d} exactly from numerator/denominator, leading to an exceedingly accurate plateau value. Thus we derived the precise plateau value h/e^2 in experiment for the principal QHE

We now extend our theory to include elementary fermions (electron, fluxon) as members of the c-fermion set. We can then treat the superconductivity and the QHE in a unified manner. The c-boson containing a pair of one electron and one fluxon can be used to describe the room temperature QHE in graphene [14].

Important pairings and effects are listed below.

(a) a pair of conduction electrons, superconductivity

(b) fluxon and c-fermions, QHE

(c) a pair of like-charge conduction electrons with two fluxons, QHE in graphene.

3. Fractional Quantum Hall Effect

We postulate that *any* c-fermion has the effective charge e^* equal to the electron charge (magnitude) e:

$$e^* = e \quad \text{for } \textit{any} \text{ c-fermion.} \tag{12}$$

Let us consider a c-fermion containing an electron and two fluxons. We shall show that the c-fermion density $n_\phi^{(2)}$ is connected with the electron density n_e by

$$n_\phi^{(2)} = n_e/2 \,. \tag{13}$$

If the c-fermions run in the E-field direction with the drift velocity v_d, the current density j is given by

$$j = e^* n_e v_d \,. \tag{14}$$

The Hall field E_H is

$$E_H = v_d B \,. \tag{15}$$

The Hall conductivity σ_H is defined and calculated as follows.

$$
\begin{aligned}
\sigma_H \;\equiv\; & \frac{j}{E_H} = \frac{e^* n_0 v_d}{v_d B} = \frac{e^* n_e}{n_\phi^{(2)}(h/e)} \\
= \; & e^* e \frac{n_e}{n_\phi^{(2)}(h/e)} = \frac{e^* e}{h} \frac{n_e}{n_\phi^{(2)}} = \frac{2e^2}{h} \,,
\end{aligned}
\tag{16}
$$

which is the correct value of σ_H at $\nu = 1/2$. The last two members of Eqs .(16) signifies Eq.(13). In fact if we assume Eqs. (12) and (13), then we obtain Equations (16).

Similarly for the case of c-fermions with four fluxons we obtain $\sigma_H = 4e^2/h$.

Extending Eq.(13) to a general case, we obtain

$$n_\phi^{(m)} = n_e/m \,, \tag{17}$$

where m is an odd number. The density $n_\phi^{(m)}$ is proportional to the magnetic field B. Equation (17) is valid for a large m. As the magnetic field is raised, the separation between the LL becomes greater, and the higher-m c-fermion is more difficult to form energetically. This condition is unlikely to depend on

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the statistics of the c-particles. Thus Eq. (17) should be valid for all integers, odd or even.

We take the case of $m = 3$. The c-boson containing an electron and three fluxons can be formed from a c-fermion with two fluxons and a fluxon by the phonon exchange attraction. If the c-bosons are Bose-condensed, the supercurrent density j is given by Eq. (9). Hence we obtain

$$
\begin{aligned}
\rho_{\mathrm{H}} &\equiv \frac{E_{\mathrm{H}}}{j} = \frac{v_{\mathrm{d}} B}{e^* n_0 v_{\mathrm{d}}} = \frac{n_\phi^{(3)}}{e^* n_0} \left(\frac{h}{e} \right) \\
&= \frac{1}{3} \frac{h}{e^2},
\end{aligned}
\tag{18}
$$

where we used $e^* = e$ from Eq. (12), and $n_\phi^{(3)}/n_0 = 1/3$, a bosonic extension of Eq. (17).

The principal fractional QHE occurs at $\nu = 1/3$, where the Hall resistivity value is $h/3e^2$. A set of weaker QHE occur on the weaker field side at

$$
\nu = 1/3, \ 2/3, \ \cdots . \tag{19}
$$

The QHE behavior at $\nu = P/Q$ for any Q is similar. We illustrate it by taking integer QHE with $\nu = P = 1, \ 2, \ \cdots$. The field magnitude becomes smaller with increasing P. The LL degeneracy is proportional to B, and hence P LL's must be considered. First consider the case $P = 2$. Without the phonon-exchange attraction the electrons occupy the lowest two LL's with spin. The electrons at each level form fundamental (f) c-bosons. In the

electron⬡───────   c-boson⬡ ▭▭▭▭
                                              ⟩ ε_g
        ◯─────   ⬡ ▭▭▭▭
          Landau levels   Phonon   ⟩
          for electrons   Exchange
                                   fc–boson energy
                                   bands with the gap ε_g
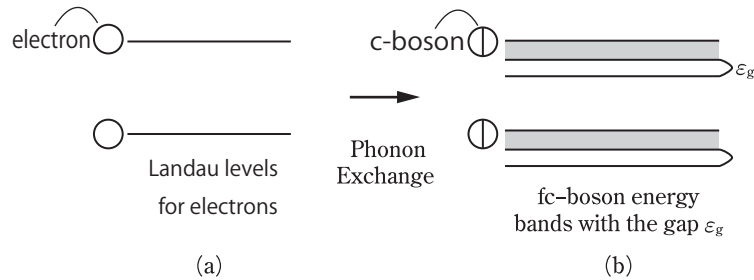   (a)   (b)

Figure 1: The electrons which fill up the lowest two LL's, form the QH state at $\nu = 2$ after the phonon-exchange attraction and the BEC of the c-bosons.

superconducting state the supercondensate occupy the monochromatically

condensed state, which is separated by the superconducting gap ε_{g} from the continuum states (band) as shown in the right-hand figure in Fig. 1.

The c-boson density n_0 at each LL is one-half the density at $\nu = 1$, which is equal to the electron density n_{e} fixed for the sample. Extending the theory to a general integer, we have

$$n_0 = n_{\mathrm{e}}/P. \tag{20}$$

This means that both $T_{\mathrm{c}} (\propto n_0^{1/2})$ and ε_{g} are smaller, making the plateau width (a measure of ε_{g}) smaller in agreement with experiments. The c-bosons have lower energies than the conduction electrons. Hence at the extreme low temperatures the supercurrent due to the condensed c-bosons dominates the normal current due to the conduction electrons and non-condensed c-bosons, giving rise to the dip in ρ.

4. Summary and Discussion

In summary we have achieved our goal of treating fractional QHE in terms of c-particles without using Laughlin's results in terms of fractional charges carried by the quasiparticles. We found that the principal fractional QHE occurs at $\nu = 1/Q$, Q (odd integers) with the Hall conductivity $\sigma_{\mathrm{H}} = Q e^2/h$, where the density of c-bosons with Q fluxons, $n_\phi^{(Q)}$, is connected with the density of condensed c-bosons with Q fluxons, n_0, by $n_\phi^{(Q)} = n_0/Q$.

Other significant findings are:

- A set of weaker fractional QHE occur on the smaller field side at $\nu = P/Q$, with the Hall conductivity $\sigma_{\mathrm{H}} = (Q/P) e^2/h$. The plateau widths (a measure of ε_{g}) are smaller are since the condensed c-boson density n_0 is smaller. Both the critical temperature $T_{\mathrm{c}} (\propto n_0^{1/2})$ and the superconducting energy gap ε_{g} are smaller.

- The fractional fermionic principal QHE with a finite conductivity and the Hall conductivity σ_{H} equal to $Q (e^2/h)$, occur at $\nu = 1/Q$, Q (even integers), where the density of c-fermions with Q fluxons, $n_\phi^{(Q)}$, is connected with the density of conduction electrons, n_{e}, by $n_\phi^{(Q)} = n_{\mathrm{e}}/Q$. All of the QHE points fall on the straight line representing the classical Hall effect if the σ_{H} is plotted as a function of $\nu = 1/Q$.

- The higher in m c-particle is more difficult to form energetically since the LL separation is greater.

In the previous work [14] we showed that

- the horizontal stretch of σ_H accompanied by the zero resistivity at the mid-point of the stretch, the signature of the QHE, arises from the superconducting energy gap ε_g.

- The cause of the QHE is the phonon exchange attraction between c-fermion and fluxons. This allows us to develop a unified theory of superconductivity and QHE.

- The Hall conductivity σ_H ($\equiv$ the current density j/the Hall field E_H) can be calculated exactly with the assumption of the BEC of the c-bosons.

Our quantum statistical theory is a finite temperature theory. The room-temperature QHE in graphene is an important topic, which is discussed separately. See Ref. [14].

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